

# FUNDAMENTAL GROUP OF DESARGUES CONFIGURATION SPACES

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**ABSTRACT.** We compute the fundamental group of various spaces of Desargues configurations in complex projective spaces: planar and non-planar configurations, with a fixed center and also with an arbitrary center.

## 1. INTRODUCTION

Let  $M$  be a manifold and  $\mathcal{F}_k(M)$  be its *ordered configuration space* of  $k$ -tuples  $\{(x_1, \dots, x_k) \in M^k \mid x_i \neq x_j, i \neq j\}$ . The  $k^{\text{th}}$  *pure braid group* of  $M$  is the fundamental group of  $\mathcal{F}_k(M)$ . The pure braid group of the plane, denoted by  $\mathcal{PB}_n$ , has the presentation [4]

$$\pi_1(\mathcal{F}_n(\mathbb{C})) = \mathcal{PB}_n \cong \langle \alpha_{ij}, 1 \leq i < j \leq n \mid (YB3)_n, (YB4)_n \rangle$$

where generators  $\alpha_{ij}$  are represented in the figure and the Yang-Baxter relations

$$\alpha_{ij} \quad \begin{array}{cccccccc} 1 & i-1 & i & i+1 & j-1 & j & j+1 & n \\ \left| \right. & \cdots & \left| \right. & \left| \right. & \left| \right. & \left| \right. & \left| \right. & \left| \right. \\ \left. \right| & & \left. \right| & \left. \right| & \left. \right| & \left. \right| & \left. \right| & \left. \right| \\ 1 & i-1 & i & i+1 & j-1 & j & j+1 & n \end{array}$$

$(YB3)_n$  and  $(YB4)_n$  are, for any  $1 \leq i < j < k \leq n$ ,

$$(YB3)_n : \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik}$$

and, for any  $1 \leq i < j < k < l \leq n$ ,

$$(YB4)_n : [\alpha_{kl}, \alpha_{ij}] = [\alpha_{jl}, \alpha_{ik}^{-1}\alpha_{ik}\alpha_{jk}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}] = 1.$$

The pure braid group of  $S^2 \approx \mathbb{CP}^1$  have the presentation (see [5] and [4]):

$$\pi_1(\mathcal{F}_{k+1}(S^2)) \cong \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB3)_k, (YB4)_k, D_k^2 = 1 \rangle,$$

where  $D_k = \alpha_{12}(\alpha_{13}\alpha_{23}) \dots (\alpha_{1k} \dots \alpha_{k-1,k})$  (in  $\mathcal{B}_k$ , the Artin braid group,  $D_k$  is the square of the Garside element  $\Delta_k$ , see [6] and [2]). In [2] we started to study the topology of configuration spaces under simple geometrical restrictions. Using the geometry of the projective space we can stratify the configuration space  $\mathcal{F}_k(\mathbb{CP}^n)$  with complex submanifolds:

$$\mathcal{F}_k(\mathbb{CP}^n) = \coprod_{i=1}^n \mathcal{F}_k^{i,n},$$

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where  $\mathcal{F}_k^{i,n}$  is the ordered configuration space of all  $k$ -tuples in  $\mathbb{CP}^n$  generating a subspace of dimension  $i$ . Their fundamental groups are given by (see [2]):

**Theorem 1.1.** *The spaces  $\mathcal{F}_k^{i,n}$  are simply connected with the following exceptions*

(1) for  $k \geq 2$ ,

$$\pi_1(\mathcal{F}_{k+1}^{1,1}) \cong \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB3)_k, (YB4)_k, D_k^2 = 1 \rangle;$$

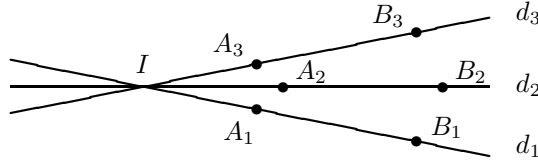
(2) for  $k \geq 3$  and  $n \geq 2$ ,

$$\pi_1(\mathcal{F}_{k+1}^{1,n}) \cong \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB3)_k, (YB4)_k, D_k = 1 \rangle.$$

In this paper we compute the fundamental groups of various configuration spaces related to projective Desargues configurations. We do not use special notations for the dual projective space: if  $P_1, P_2, P_3$  are three points and  $d_1, d_2, d_3$  are three lines in  $\mathbb{CP}^2$ ,  $(P_1, P_2, P_3) \in \mathcal{F}_3^{1,2}$  is equivalent with the collinearity of these points and  $(d_1, d_2, d_3) \in \mathcal{F}_3^{1,2}$  is equivalent with the concurrency of these lines. We define  $\mathcal{D}^{2,n}$ , the space of planar Desargues configurations in  $\mathbb{CP}^n$  ( $n \geq 2$ ), by

$$\mathcal{D}^{2,n} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}_6^{2,n} \mid (d_1, d_2, d_3) \in \mathcal{F}_3^{1,2}, A_i, B_i \in d_i \setminus \{I\}\}$$

(here  $I = d_1 \cap d_2 \cap d_3$ ).



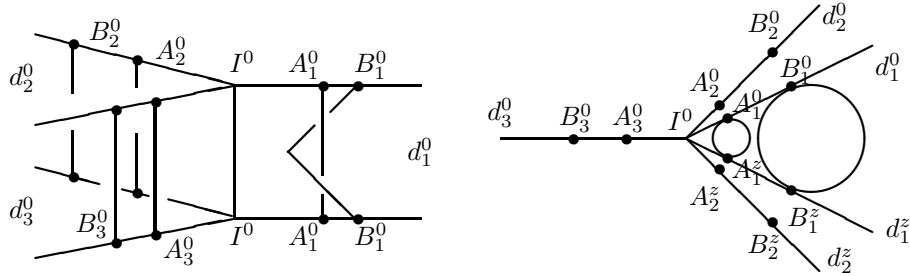
We consider also  $\mathcal{D}_I^{2,n}$ , the space of planar Desargues configuration with a fixed intersection point  $I \in \mathbb{CP}^n$ , defined by

$$\mathcal{D}_I^{2,n} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{D}^{2,n} \mid d_1 \cap d_2 \cap d_3 = I\}.$$

**Theorem 1.2.** *The fundamental group of  $\mathcal{D}_I^{2,n}$  is given by*

$$\pi_1(\mathcal{D}_I^{2,n}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \geq 3. \end{cases}$$

The first group is generated by  $[\alpha]$ ,  $[\beta]$ , and  $[\sigma]$ , and the second group is generated by  $[\alpha]$  and  $[\beta]$ . Precise formulae for  $\alpha, \beta$  and  $\sigma$  are given in section 2; here is a diagram representing these generators (there is a similar picture for  $\beta$ ):



$\alpha$  :  $B_1$  is moving on the line  $d_1^0 \setminus \{I^0, A_1^0\}$

$\sigma$  : the lines  $d_1$  and  $d_2$  are moving

**Theorem 1.3.** *The fundamental group of  $\mathcal{D}^{2,n}$  is given by:*

$$\pi_1(\mathcal{D}^{2,n}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z} & \text{if } n \geq 3. \end{cases}$$

The first group is generated by  $[\alpha]$  and  $[\beta]$  and the second group is generated by  $[\alpha]$  (or by  $[\beta]$ ); we will use the same notations for  $[\alpha]$ ,  $[\beta]$ ,  $[\sigma]$  and their images through different natural maps:  $\mathcal{D}_I^{*,*} \rightarrow \mathcal{D}^{*,*}$ ,  $\mathcal{D}_I^{*,*} \rightarrow \mathcal{D}_I^{*,*+1}$ ,  $\mathcal{D}^{*,*} \rightarrow \mathcal{D}^{*,*+1}$ .

We define  $\mathcal{D}^{3,n}$ , the *space of non-planar Desargues configurations in  $\mathbb{CP}^n$  ( $n \geq 3$ )*:

$$\mathcal{D}^{3,n} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}_6^{3,n} \mid d_1 \cap d_2 \cap d_3 = I, A_i, B_i \in d_i \setminus \{I\}\}$$

and  $\mathcal{D}_I^{3,n}$ , the *associated space of non-planar Desargues configurations with a fixed intersection point  $I \in \mathbb{CP}^n$* .

**Theorem 1.4.** *The fundamental group of  $\mathcal{D}_I^{3,n}$  is given by:*

$$\pi_1(\mathcal{D}_I^{3,n}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 3, \\ 1 & \text{if } n \geq 4. \end{cases}$$

**Theorem 1.5.** *The fundamental group of  $\mathcal{D}^{3,n}$  is given by:*

$$\pi_1(\mathcal{D}^{3,n}) \cong \begin{cases} \mathbb{Z}_4 & \text{if } n = 3, \\ 1 & \text{if } n \geq 4. \end{cases}$$

In the last two theorems, in the non-simply connected cases, the fundamental groups are generated by  $[\alpha]$ .

## 2. DESARGUES CONFIGURATIONS IN THE PROJECTIVE PLANE

In order to find the fundamental groups of the spaces  $\mathcal{D} = \mathcal{D}^{2,2}$  and  $\mathcal{D}_I = \mathcal{D}_I^{2,2}$  we use two fibrations and their homotopy exact sequences.

**Lemma 2.1.** *The projection*

$$\mu : \mathcal{D} \rightarrow \mathbb{CP}^2, (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto I = d_1 \cap d_2 \cap d_3$$

*is a locally trivial fibration with fiber  $\mathcal{D}_I$ .*

*Proof.* Fix a point  $I^0 \in \mathbb{CP}^2$  and choose a line  $l \subset \mathbb{CP}^2 \setminus \{I^0\}$  and the neighborhood  $\mathcal{U}_l = \mathbb{CP}^2 \setminus l$  of  $I^0$ . For a point  $I$  in this neighborhood and a Desargues configuration  $(A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)$  on three lines  $d_1^0, d_2^0, d_3^0$  containing  $I^0$  construct lines  $d_1, d_2, d_3$  containing  $I$  and the configuration  $(A_1, B_1, \dots, A_3, B_3)$  as follows: consider the points  $D_i = l \cap d_i^0$  and  $Q = l \cap I^0 I$  and define  $d_i = ID_i$ ,  $A_i = d_i \cap QA_i^0$  and in the same way  $B_i$  ( $i = 1, 2, 3$ ). We describe this construction using coordinates to show that the map

$$(I, (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)) \mapsto (A_1, B_1, A_2, B_2, A_3, B_3)$$

has a continuous extension on the singular locus  $(d_1^0 \cup d_2^0 \cup d_3^0 \setminus l)$ . Choose a projective frame such that  $I^0 = [0 : 0 : 1]$ ,  $l : X_2 = 0$ . If  $I = [s : t : 1]$  and  $A_i^0 = [n_i : -m_i : a_i]$ ,  $B_i^0 = [n_i : -m_i : b_i]$  ( $a_i, b_i$  are distinct and non zero and also  $n_i m_j \neq m_i n_j$  for distinct  $i, j = 1, 2, 3$ ), then we define  $A_i = [n_i + sa_i : -m_i + ta_i : a_i]$  and  $B_i = [n_i + sb_i : -m_i + tb_i : b_i]$ , ( $i = 1, 2, 3$ ), and these formulae agree with the geometrical construction given for nondegenerate positions of  $I \in \mathbb{CP}^2 \setminus (d_1^0 \cup d_2^0 \cup d_3^0 \cup l)$ . The trivialization over  $\mathcal{U}_l$  is given by

$$\varphi : \mathcal{U}_l \times \mathcal{D}_{I^0} \rightarrow \gamma^{-1}(\mathcal{U}_l), \varphi(I, (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)) = (A_1, B_1, A_2, B_2, A_3, B_3).$$

□

**Lemma 2.2.** *The projection*

$$\lambda : \mathcal{D}_I \rightarrow \mathcal{F}_3(\mathbb{CP}^1), (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (d_1, d_2, d_3)$$

*is a locally trivial fibration with fiber  $\mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C})$ .*

*Proof.* Fix a point  $d_*^0 = (d_1^0, d_2^0, d_3^0)$  in  $\mathcal{F}_3(\mathbb{CP}^1)$  and choose a point  $Q$  in  $\mathbb{CP}^2 \setminus (d_1^0 \cup d_2^0 \cup d_3^0)$  and the neighborhood  $\mathcal{U}_Q = \{(d_1, d_2, d_3) \in \mathcal{F}_3(\mathbb{CP}^1) \mid Q \notin d_1 \cup d_2 \cup d_3\}$ . The trivialization over  $\mathcal{U}_Q$  is given by

$$\psi : \mathcal{U}_Q \times \mathcal{F}_2(d_1^0 \setminus \{I\}) \times \mathcal{F}_2(d_2^0 \setminus \{I\}) \times \mathcal{F}_2(d_3^0 \setminus \{I\}) \rightarrow \lambda^{-1}(\mathcal{U}_Q)$$

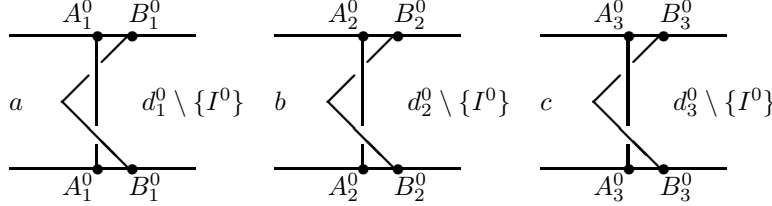
$$\psi((d_1, d_2, d_3), (A_1^0, B_1^0), (A_2^0, B_2^0), (A_3^0, B_3^0)) = (A_1, B_1, A_2, B_2, A_3, B_3),$$

where  $A_i = d_i \cap QA_i^0$  and similarly for  $B_i$  ( $i = 1, 2, 3$ ). Obviously,  $A_i, B_i$  and  $I$  are three distinct points on  $d_i$ .  $\square$

In  $\mathcal{D}_{I^0=[0:0:1]}$  we choose the base point  $D^0 = (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)$  where, for  $k = 1, 2$ ,  $A_k^0 = [-1 : k : 1]$ ,  $B_k^0 = [-1 : k : 2]$ ,  $A_3^0 = [0 : 1 : 1]$ ,  $B_3^0 = [0 : 1 : 2]$ . The corresponding lines are given by the equations  $d_k^0 : kX_0 + X_1 = 0$ ,  $d_3^0 : X_0 = 0$  and we identify the affine line  $\mathbb{C}$  with  $d_k^0$  as follows: for  $k = 1, 2$ ,  $z \mapsto [-1 : k : z]$ , and for  $k = 3$ ,  $z \mapsto [0 : 1 : z]$  (therefore the intersection point  $I^0 = [0 : 0 : 1]$  is the point at infinity of these lines). We identify the set of three distinct lines through  $I^0$  with the configuration space  $\mathcal{F}_3(\mathbb{CP}^1)$ ; in this space the base point is  $d_*^0 = (d_1^0, d_2^0, d_3^0)$ . In the configuration spaces  $\mathcal{F}_2(d_i^0 \setminus \{I^0\})$  we choose the base points  $(A_i^0, B_i^0)$ ,  $i = 1, 2, 3$ . The homotopy exact sequence from Lemma 2.2 and the triviality of  $\pi_2(\mathcal{F}_3(\mathbb{CP}^1))$  (see [3]) give the short exact sequence

$$1 \rightarrow \pi_1(\mathcal{F}_2(\mathbb{C})) \times \pi_1(\mathcal{F}_2(\mathbb{C})) \times \pi_1(\mathcal{F}_2(\mathbb{C})) \xrightarrow{j_*} \pi_1(\mathcal{D}_{I^0}) \xrightarrow{\lambda_*} \pi_1(\mathcal{F}_3(\mathbb{CP}^1)) \rightarrow 1.$$

*Proof of Theorem 1.2 (the case  $n = 2$ ).* The first group, isomorphic to  $\mathbb{Z}^3$ , is generated by the pure braids  $a, b, c$ , hence their images in  $\pi_1(\mathcal{D}_{I^0})$  are given by the



homotopy classes of the maps  $\alpha, \beta, \gamma : (S^1, 1) \rightarrow (\mathcal{D}_{I^0}, D^0)$

$$\begin{aligned} \alpha(z) &= (A_1^0, B_1^{\alpha(z)}, A_2^0, B_2^0, A_3^0, B_3^0), & B_1^{\alpha(z)} &= [-1 : 1 : 1 + z], \\ \beta(z) &= (A_1^0, B_1^0, A_2^0, B_2^{\beta(z)}, A_3^0, B_3^0), & B_2^{\beta(z)} &= [-1 : 2 : 1 + z], \\ \gamma(z) &= (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^{\gamma(z)}), & B_3^{\gamma(z)} &= [0 : 1 : 1 + z]. \end{aligned}$$

The third group,  $\pi_1(\mathcal{F}_3(\mathbb{CP}^1)) \cong \mathbb{Z}_2$ , is generated by the homotopy class of the map

$$s : (S^1, 1) \rightarrow (\mathcal{F}_3(\mathbb{CP}^1), d_*^0), z \mapsto (d_1^{s(z)} : zX_0 + X_1 = 0, d_2^{s(z)} : 2zX_0 + X_1 = 0, d_3^0),$$

because this corresponds to the braid  $\alpha_{12}$  in  $\mathbb{CP}^1$ . We lift the map  $s$  to the map

$$\sigma : (S^1, 1) \rightarrow (\mathcal{D}_I^0, D^0), z \mapsto (A_1^{\sigma(z)}, B_1^{\sigma(z)}, A_2^{\sigma(z)}, B_2^{\sigma(z)}, A_3^0, B_3^0),$$

where  $A_k^{\sigma(z)} = [-1 : kz : 1]$ ,  $B_k^{\sigma(z)} = [-1 : kz : 2]$ ,  $k = 1, 2$ .

The group  $\pi_1(\mathcal{D}_{I^0}, D^0)$  is generated by the homotopy classes of  $\alpha, \beta, \gamma$  and  $\sigma$ ; the defining relations are commutation relations between  $[\alpha], [\beta]$  and  $[\gamma]$  from  $\pi_1(\mathcal{F}_2(\mathbb{C})^3)$  and the four relations, to be proved in the next two lemmas:

$$\begin{aligned} \alpha) \quad & [\sigma][\alpha][\sigma]^{-1} = [\alpha], \\ \beta) \quad & [\sigma][\beta][\sigma]^{-1} = [\beta], \\ \gamma) \quad & [\sigma][\gamma][\sigma]^{-1} = [\gamma], \\ \sigma) \quad & [\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma]. \end{aligned}$$

The generator  $[\gamma]$  can be eliminated,  $[\sigma]$  commutes with  $[\alpha]$  and  $[\beta]$ , and the third relation,  $\gamma)$ , is a consequence of the previous commutation relations.  $\square$

**Lemma 2.3.** *In  $\pi_1(\mathcal{D}_{I^0}, D^0)$  the next relation holds:*

$$\sigma) \quad [\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma].$$

*Proof.* The map

$$\Lambda : (D^2, S^1) \rightarrow (\mathcal{F}_3(\mathbb{CP}^1), d_*^0 = (d_1^0, d_2^0, d_3^0)), \quad z \mapsto (d_1^{\Lambda(z)}, d_2^{\Lambda(z)}, d_3^{\Lambda(z)}),$$

where  $d_k^{\Lambda(z)} : (kz - r)X_0 + (\bar{z} + kr)X_1 = 0$ , ( $k = 1, 2$ ), and  $d_3^{\Lambda(z)} : zX_0 + rX_1 = 0$  (the notation  $r = 1 - |z|$  will be used in this proof and the next ones), shows that  $s^2 \simeq \text{constant}_{d^0}$ . We lift this homotopy to

$$\tilde{\Lambda} : D^2 \rightarrow \mathcal{D}_{I^0}, \quad \tilde{\Lambda}(z) = (A_1^{\tilde{\Lambda}(z)}, B_1^{\tilde{\Lambda}(z)}, A_2^{\tilde{\Lambda}(z)}, B_2^{\tilde{\Lambda}(z)}, A_3^{\tilde{\Lambda}(z)}, B_3^{\tilde{\Lambda}(z)}),$$

where  $A_k^{\tilde{\Lambda}(z)} = [-\bar{z} - kr : kz - r : \bar{z}]$ ,  $B_k^{\tilde{\Lambda}(z)} = [-\bar{z} - kr : kz - r : \bar{z} + 1]$ , ( $k = 1, 2$ ), and  $A_3^{\tilde{\Lambda}(z)} = [-r : z : z]$ ,  $B_3^{\tilde{\Lambda}(z)} = [-r : z : z + 1]$ ; the map

$$\tilde{\Lambda}|_{S^1} : S^1 \rightarrow \mathcal{D}_{I^0}, \quad z \mapsto (A_1^z, B_1^z, A_2^z, B_2^z, A_3^0, B_3^z)$$

(with  $A_k^z = [-1 : kz^2 : 1]$ ,  $B_k^z = [-1 : kz^2 : 1 + z]$ ,  $k = 1, 2$ , and  $B_3^z = [0 : 1 : 1 + \bar{z}]$ ) has a trivial homotopy class, therefore we have the relation  $[\sigma]^2 = [\sigma * \sigma * (\tilde{\Lambda}|_{S^1})^{-1}]$ .

Now we construct a homotopy between  $\sigma * \sigma * (\tilde{\Lambda}|_{S^1})^{-1}$  and  $\alpha^{-1} * \beta^{-1} * \gamma$ :

$$L : S^1 \times I \rightarrow \mathcal{D}_{I^0}, \quad (z, t) \mapsto (A_1^{L(z,t)}, B_1^{L(z,t)}, A_2^{L(z,t)}, B_2^{L(z,t)}, A_3^0, B_3^{L(z,t)}),$$

where ( $k = 1, 2$ ):

$$A_k^{L(z,t)} = [-1 : kL^1(z, t) : 1], \quad B_k^{L(z,t)} = [-1 : kL^1(z, t) : L_k^2(z, t)]$$

$$B_3^{L(z,t)} = \begin{cases} [0 : 1 : 2] & 0 \leq \arg z \leq \pi \\ [0 : 1 : 1 + z^2] & \pi \leq \arg z \leq 2\pi, \end{cases}$$

and

$$\begin{aligned} L^1(z, t) &= \begin{cases} z^4 & 0 \leq \arg z \leq t\pi \\ \exp(4t\pi i) & t\pi \leq \arg z \leq (2-t)\pi \\ \bar{z}^4 & (2-t)\pi \leq \arg z \leq 2\pi, \end{cases} \\ L_k^2(z, t) &= \begin{cases} 2 & 0 \leq \arg z \leq \frac{t+k-1}{k}\pi \\ 1 + \exp\left(4\frac{(2-k)t\pi - \arg z}{1+t}i\right) & \frac{t+k-1}{k}\pi \leq \arg z \leq \frac{1+(5-2k)t}{3-k}\pi \\ 2 & \frac{1+(5-2k)t}{3-k}\pi \leq \arg z \leq 2\pi. \end{cases} \end{aligned}$$

It is easy to check that  $L(-, 0) = (\alpha^{-1} * \beta^{-1}) * \gamma$  and  $L(-, 1) = (\sigma * \sigma) * (\tilde{\Lambda}|_{S^1})^{-1}$ .  $\square$

**Lemma 2.4.** *In  $\pi_1(\mathcal{D}_{I^0}, D^0)$  the next relations hold:*

$$\begin{aligned} \alpha) \quad & [\sigma][\alpha][\sigma]^{-1} = [\alpha]; \\ \beta) \quad & [\sigma][\beta][\sigma]^{-1} = [\beta]; \\ \gamma) \quad & [\sigma][\gamma][\sigma]^{-1} = [\gamma]. \end{aligned}$$

*Proof.* The loop  $\sigma * \alpha * \sigma^{-1}$  in  $\mathcal{D}_{I^0}$  is given by  $z \mapsto (A_1^{\tilde{\alpha}(z)}, B_1^{\tilde{\alpha}(z)}, A_2^{\tilde{\alpha}(z)}, B_2^{\tilde{\alpha}(z)}, A_3^0, B_3^0)$ , where the points  $A_k^{\tilde{\alpha}(z)}$  ( $k = 1, 2$ ),  $B_1^{\tilde{\alpha}(z)}$  and  $B_2^{\tilde{\alpha}(z)}$  are given by :

$$\begin{aligned} A_k &= [-1 : kz^3 : 1] & B_1 &= [-1 : z^3 : 2] & B_2 &= [-1 : 2z^3 : 2] & \arg z &\in [0, \frac{2\pi}{3}] \\ A_k &= A_k^0 & B_1 &= [-1 : 1 : 1 + z^3] & B_2 &= B_2^0 & \arg z &\in [\frac{2\pi}{3}, \frac{4\pi}{3}] \\ A_k &= [-1 : k\bar{z}^3 : 1] & B_1 &= [-1 : \bar{z}^3 : 2] & B_2 &= [-1 : 2\bar{z}^3 : 2] & \arg z &\in [\frac{4\pi}{3}, 2\pi]. \end{aligned}$$

We define two maps

$$\begin{aligned} \varepsilon : S^1 \times I &\rightarrow S^1, \quad \varepsilon(z, t) = \begin{cases} z^3 & 0 \leq \arg z \leq \frac{2t}{3}\pi \\ \exp(2t\pi i) & \frac{2t}{3}\pi \leq \arg z \leq \frac{2(3-t)}{3}\pi \\ \bar{z}^3 & \frac{2(3-t)}{3}\pi \leq \arg z \leq 2\pi, \end{cases} \\ \eta : S^1 &\rightarrow \mathbb{C} \setminus \{1\}, \quad \eta(z) = \begin{cases} 2 & \arg z \in [0, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, 2\pi] \\ 1 + z^3 & \arg z \in [\frac{2\pi}{3}, \frac{4\pi}{3}]. \end{cases} \end{aligned}$$

and a new homotopy

$$K_\alpha(z, t) : S^1 \times I \rightarrow \mathcal{D}_{I^0}, \quad K_\alpha(z, t) = (A_1(z, t), \tilde{B}_1(z, t), A_2(z, t), B_2(z, t), A_3^0, B_3^0),$$

where  $A_k(z, t) = [-1 : k\varepsilon(z, t) : 1]$ ,  $B_k(z, t) = [-1 : k\varepsilon(z, t) : 2]$ , ( $k = 1, 2$ ),  $\tilde{B}_1(z, t) = [-1 : \varepsilon(z, t) : \eta(z)]$ . One can check that  $K_\alpha|_{t=0} \simeq \alpha$  and  $K_\alpha|_{t=1} = \sigma * \alpha * \sigma^{-1}$ . Similarly we have a homotopy  $K_\beta$  between  $\beta$  and  $K_\beta|_{t=1} = \sigma * \beta * \sigma^{-1}$ . Next homotopy (we also use the notation  $B_3(z, t) = [0 : 1 : \eta(z)]$ )

$$K_\gamma(z, t) : S^1 \times I \rightarrow \mathcal{D}_{I^0}, \quad (z, t) \mapsto (A_1(z, t), B_1(z, t), A_2(z, t), B_2(z, t), A_3^0, B_3(z, t)),$$

gives the last relation:  $K_\gamma|_{t=0} \simeq \gamma$ ,  $K_\gamma|_{t=1} = \sigma * \gamma * \sigma^{-1}$ .  $\square$

*Proof of Theorem 1.3 (the case  $n = 2$ ).* Lemma 2.1 gives the exact sequence

$$\dots \longrightarrow \pi_2(\mathbb{CP}^2) \xrightarrow{\delta_*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(\mathcal{D}) \longrightarrow 1$$

where the first group is cyclic generated by the homotopy class of the map

$$\Phi : (D^2, S^1) \rightarrow (\mathbb{CP}^2, I^0), \quad z \mapsto [0 : r : z].$$

We choose the lift

$$\tilde{\Phi} : (D^2, S^1) \rightarrow (\mathcal{D}, \mathcal{D}_{I^0}), \quad z \mapsto (A_1^{\tilde{\Phi}(z)}, B_1^{\tilde{\Phi}(z)}, A_2^{\tilde{\Phi}(z)}, B_2^{\tilde{\Phi}(z)}, A_3^{\tilde{\Phi}(z)}, B_3^{\tilde{\Phi}(z)}),$$

where ( $k = 1, 2$ )

$$\begin{aligned} A_k^{\tilde{\Phi}(z)} &= [-1 : (2k+1)r + k\bar{z} : (2k+1)z + k(r-2)], \\ B_k^{\tilde{\Phi}(z)} &= [-1 : (2k+2)r + k\bar{z} : (2k+2)z + k(r-2)], \\ A_3^{\tilde{\Phi}(z)} &= [-r : \bar{z} + 4r : 4z - 3(r+1)], \\ B_3^{\tilde{\Phi}(z)} &= [-r : \bar{z} + 5r : 5z - 3(r+1)], \end{aligned}$$

hence  $\text{Im } \delta_*$  is generated by the homotopy class of the map

$$\tilde{\Phi}|_{S^1} : S^1 \rightarrow \mathcal{D}_{I^0}, \quad z \mapsto (A_1^{\Phi(z)}, B_1^{\Phi(z)}, A_2^{\Phi(z)}, B_2^{\Phi(z)}, A_3^{\Phi(z)}, B_3^{\Phi(z)}),$$

with  $(k = 1, 2)$

$$\begin{aligned} A_k^{\Phi(z)} &= [-1 : k\bar{z} : (2k+1)z - 2k], & B_k^{\Phi(z)} &= [-1 : k\bar{z} : (2k+2)z - 2k], \\ A_3^{\Phi(z)} &= [0 : \bar{z} : 4z - 3], & B_3^{\Phi(z)} &= [0 : \bar{z} : 5z - 3]. \end{aligned}$$

The maps  $\lambda \circ \tilde{\Phi}|_{S^1}$  and  $s^{-1}$  coincide, therefore the product  $[\tilde{\Phi}|_{S^1}] \cdot [\sigma]$  belongs to  $\ker \lambda_* = \text{Im } j_*$ . We show that  $[\tilde{\Phi}|_{S^1}] \cdot [\sigma] = [\alpha] \cdot [\beta] \cdot [\gamma]$  and this implies the claim of the theorem. We define the homotopy:

$$H : S^1 \times I \rightarrow \mathcal{D}_{I^0}, (z, t) \mapsto (A_1^{H(z,t)}, B_1^{H(z,t)}, A_2^{H(z,t)}, B_2^{H(z,t)}, A_3^{H(z,t)}, B_3^{H(z,t)}),$$

where  $(k = 1, 2)$

$$\begin{aligned} A_k^{H(z,t)} &= [-1 : H_k^1(z, t) : H_k^2(z, t)] & B_k^{H(z,t)} &= [-1 : H_k^1(z, t) : H_k^2(z, t) + H_k^4(z, t)] \\ A_3^{H(z,t)} &= [0 : 1 : H^3(z, t)] & B_3^{H(z,t)} &= [0 : 1 : H^3(z, t) + H^5(z, t)] \end{aligned}$$

and

$$\begin{aligned} H_k^1(z, t) &= \begin{cases} k\bar{z}^2 & 0 \leq \arg z \leq t\pi \\ k \exp(-2t\pi i) & t\pi \leq \arg z \leq (2-t)\pi \\ k\bar{z}^2 & (2-t)\pi \leq \arg z \leq 2\pi, \end{cases} \\ H_k^2(z, t) &= \begin{cases} 1 + (2k+1)t(z^2 - 1) & 0 \leq \arg z \leq \pi \\ 1 & \pi \leq \arg z \leq 2\pi, \end{cases} \\ H^3(z, t) &= \begin{cases} 1 + t(4z^4 - 3z^2 - 1) & 0 \leq \arg z \leq \pi \\ 1 & \pi \leq \arg z \leq 2\pi, \end{cases} \\ H_1^4(z, t) &= \begin{cases} \exp\left(\frac{4\arg z}{1+t}i\right) & 0 \leq \arg z \leq \frac{1+t}{2}\pi \\ 1 & \frac{1+t}{2}\pi \leq \arg z \leq 2\pi, \end{cases} \\ H_2^4(z, t) &= \begin{cases} 1 & 0 \leq \arg z \leq \frac{1-t}{2}\pi \\ \exp\left(2\frac{2\arg z - (1-t)\pi}{1+t}i\right) & \frac{1-t}{2}\pi \leq \arg z \leq \pi \\ 1 & \pi \leq \arg z \leq 2\pi. \end{cases} \\ H^5(z, t) &= \begin{cases} 1 & 0 \leq \arg z \leq (1-t)\pi \\ \exp[4(\arg z - (1-t)\pi)i] & (1-t)\pi \leq \arg z \leq (2-t)\pi \\ 1 & (2-t)\pi \leq \arg z \leq 2\pi. \end{cases} \end{aligned}$$

These computations give  $\text{Im } \delta_* = \mathbb{Z}\langle 2[\alpha] + 2[\beta] + [\sigma] \rangle$ , therefore we can choose  $[\alpha]$  and  $[\beta]$  as generators of the fundamental group of  $\mathcal{D}$ .  $\square$

### 3. PLANAR DESARGUES CONFIGURATION IN $\mathbb{CP}^n$

First we reduce the computations of  $\pi_1(\mathcal{D}_I^{2,n})$  and of  $\pi_1(\mathcal{D}^{2,n})$  to the case  $n = 3$ .

**Lemma 3.1.** *The following projections are locally trivial fibrations:*

- a)  $\mathcal{D}_I^{2,2} \hookrightarrow \mathcal{D}_I^{2,n} \rightarrow \text{Gr}^1(\mathbb{CP}^{n-1})$ ,  $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto \text{line}(d_1, d_2, d_3)$ ;
- b)  $\mathcal{D}^{2,2} \hookrightarrow \mathcal{D}^{2,n} \rightarrow \text{Gr}^2(\mathbb{CP}^n)$ ,  $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto 2\text{-plane}(d_1, d_2, d_3)$ .

*Proof.* a) Fix a 2-plane  $P_0$  through  $I$  and choose a hyperplane  $H \subset \mathbb{CP}^n$  such that  $I \notin H$  and an  $(n-3)$  dimensional subspace  $Q \subset H$  such that  $Q \cap l_0 = \emptyset$ , where  $l_0 = P_0 \cap H$ . Take as a neighborhood of  $P_0$  the set  $\{P \text{ a 2-plane in } \mathbb{CP}^n \mid I \in P, P \cap Q = \emptyset\}$  and associate to a Desargues configuration in  $\mathcal{D}_I(P_0)$  the projection from  $Q$ , an element in  $\mathcal{D}_I(P)$ :  $C_i^0 = d_i^0 \cap l_0$ ,  $l = P \cap H$ ,  $C_i = (Q \vee C_i^0) \cap l$ ,  $Q_i = Q \cap (C_i C_i^0)$ ,  $d_i = IC_i$ ,  $A_i = Q_i A_i^0 \cap d_i$ ,  $B_i = Q_i B_i^0 \cap d_i$  (for  $i = 1, 2, 3$ ). Using projective coordinates one can show that this trivialization is well defined on the

singular locus  $P = P_0$ : if  $I = [0 : \dots : 0 : 1]$ ,  $P_0 : X_0 = \dots = X_{n-3} = 0$ ,  $A_i^0 = [0 : \dots : a_{n-2,i}^0 : a_{n-1,i}^0 : a_{n,i}^0]$ ,  $B_i^0 = [0 : \dots : b_{n-2,i}^0 : b_{n-1,i}^0 : b_{n,i}^0]$ , and  $P$  is defined by the equations  $X_k = p_{k,1}X_{n-2} + p_{k,2}X_{n-1} + p_{k,3}X_n$  ( $k = 0, \dots, n-3$ ), then  $A_i = [p_{0,0}a_{n-2,i} + p_{0,1}a_{n-1,i} : \dots : p_{n-3,0}a_{n-2,i} + p_{n-3,1}a_{n-1,i} : a_{n-2,i}^0 : a_{n-1,i}^0 : a_{n,i}^0]$ ,  $B_i = [p_{0,0}b_{n-2,i} + p_{0,1}b_{n-1,i} : \dots : p_{n-3,0}b_{n-2,i} + p_{n-3,1}b_{n-1,i} : b_{n-2,i}^0 : b_{n-1,i}^0 : b_{n,i}^0]$ .

b) Fix a 2-plane  $P_0$  and choose as center of projection a disjoint  $n-3$  dimensional subspace  $Q$ . Take as a neighborhood of  $P_0$  the set of 2-planes disjoint from  $Q$ . The projection from  $Q$  associate to a Desargues configuration in  $\mathcal{D}^2(P_0)$  a Desargues configuration in  $\mathcal{D}^2(P) : P \cap (Q \vee I^0) = I$ ,  $P \cap (Q \vee d_i^0) = d_i$ ,  $d_i \cap (Q \vee A_i^0) = A_i$ ,  $d_i \cap (Q \vee B_i^0) = B_i$ .  $\square$

**Corollary 3.2.** *For  $n \geq 3$  we have*

- a)  $\pi_1(\mathcal{D}_I^{2,3}) \cong \pi_1(\mathcal{D}_I^{2,n})$ ;
- b)  $\pi_1(\mathcal{D}^{2,3}) \cong \pi_1(\mathcal{D}^{2,n})$ .

*Proof.* This is a consequence of the stability of the second homotopy group of the complex Grassmannians:

$$\begin{array}{ccccccc} \pi_2(\mathrm{Gr}^1(\mathbb{CP}^2)) & \longrightarrow & \pi_1(\mathcal{D}_I^{2,2}) & \longrightarrow & \pi_1(\mathcal{D}_I^{2,3}) & \longrightarrow & 1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \pi_2(\mathrm{Gr}^1(\mathbb{CP}^{n-1})) & \longrightarrow & \pi_1(\mathcal{D}_I^{2,2}) & \longrightarrow & \pi_1(\mathcal{D}_I^{2,n}) & \longrightarrow & 1 \end{array}$$

and also

$$\begin{array}{ccccccc} \pi_2(\mathrm{Gr}^2(\mathbb{CP}^3)) & \longrightarrow & \pi_1(\mathcal{D}^{2,2}) & \longrightarrow & \pi_1(\mathcal{D}^{2,3}) & \longrightarrow & 1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \pi_2(\mathrm{Gr}^2(\mathbb{CP}^n)) & \longrightarrow & \pi_1(\mathcal{D}^{2,2}) & \longrightarrow & \pi_1(\mathcal{D}^{2,n}) & \longrightarrow & 1. \end{array}$$

$\square$

Using the fibration of Lemma 3.1 a) for  $n = 3$  we have the exact sequence

$$\dots \rightarrow \pi_2(\mathbb{CP}^2) \xrightarrow{\delta_*} \pi_1(\mathcal{D}_I^{2,2}) \rightarrow \pi_1(\mathcal{D}_I^{2,3}) \rightarrow 1.$$

We choose the base point in  $\mathcal{D}_I^{2,3}$  the image of the base point in  $\mathcal{D}_I$  through the embedding  $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : x_2 : 0]$  and we denote the compositions  $\alpha, \beta : S^1 \rightarrow \mathcal{D}_I^{2,2} \rightarrow \mathcal{D}_I^{2,3}$  with the same letters.

**Proposition 3.3.** *In the exact sequence of the fibration  $\mathcal{D}_I^{2,3} \rightarrow \mathbb{CP}^2$  we have:*

- a)  $\mathrm{Im} \delta_* = \mathbb{Z}([\alpha] + [\beta] + [\sigma])$ ;
- b)  $\pi_1(\mathcal{D}_I^{2,3}) \cong \mathbb{Z} \oplus \mathbb{Z}$  is generated by  $[\alpha]$  and  $[\beta]$ .

*Proof.* a) The base point in  $\mathrm{Gr}^1(\mathbb{CP}^2) \approx \mathbb{CP}^2$  is the line  $X_3 = 0$  (in the dual space of lines through  $I^0 = [0 : 0 : 1 : 0]$ ) and we choose the generator of  $\pi_2(\mathbb{CP}^2)$  the homotopy class of the map

$$\Pi : (D^2, S^1) \rightarrow \mathrm{Gr}^1(\mathbb{CP}^2), z \mapsto (1 - |z|)X_1 + zX_3 = 0.$$



The lift  $\tilde{\Pi} : D^2 \rightarrow \mathcal{D}_{I^0}^{2,3}$ ,  $z \mapsto (A_1^{\tilde{\Pi}(z)}, B_1^{\tilde{\Pi}(z)}, A_2^{\tilde{\Pi}(z)}, B_2^{\tilde{\Pi}(z)}, A_3^{\tilde{\Pi}(z)}, B_3^{\tilde{\Pi}(z)})$  is given by ( $k = 1, 2$ )

$$\begin{aligned} A_k^{\tilde{\Pi}(z)} &= [2r|z| - 1 : kz : 1 : -kr], & A_3^{\tilde{\Pi}(z)} &= [0 : z : z : -r], \\ B_k^{\tilde{\Pi}(z)} &= [2r|z| - 1 : kz : 2 : -kr], & B_3^{\tilde{\Pi}(z)} &= [0 : z : z + 1 : -r], \end{aligned}$$

where the corresponding lines are

$$d_k^{\tilde{\Pi}(z)} : kX_0 + \bar{z}X_1 - rX_3 = 0, \quad rX_1 + zX_3 = 0, \quad d_3^{\tilde{\Pi}(z)} : X_0 = 0, \quad rX_1 + zX_3 = 0.$$

The homotopy

$$M : S^1 \times I \rightarrow \mathcal{D}_{I^0}^{2,2}, \quad (z, t) \mapsto (A_1^{M(z,t)}, B_1^{M(z,t)}, A_2^{M(z,t)}, B_2^{M(z,t)}, A_3^0, B_3^{M(z,t)}),$$

where  $A_k^{M(z,t)} = [-1 : km_1(z, t) : 1]$ ,  $B_k^{M(z,t)} = [-1 : km_1(z, t) : 2]$ , and  $B_3^{M(z,t)} = [0 : 1 : 1 + m_2(z, t)]$  are defined by:

$$\begin{aligned} m_1(z, t) &= \begin{cases} \exp\left(2\frac{\arg z}{2-t}i\right) & 0 \leq \arg z \leq (2-t)\pi \\ 1 & (2-t)\pi \leq \arg z \leq 2\pi, \end{cases} \\ m_2(z, t) &= \begin{cases} 1 & 0 \leq \arg z \leq t\pi \\ \exp\left(2\frac{t\pi - \arg z}{2-t}i\right) & t\pi \leq \arg z \leq 2\pi, \end{cases} \end{aligned}$$

shows that the restriction  $\tilde{\Pi}|_{S^1}$  and the loop  $\sigma * \gamma^{-1}$  are homotopic. Using this and the relation  $[\gamma] = [\alpha] + [\beta] + 2[\sigma]$  we find  $\delta_*([\Pi]) = [\tilde{\Pi}|_{S^1}] = -[\alpha] - [\beta] - [\sigma]$ .

b) The second part is a consequence of part a).  $\square$

**Proposition 3.4.** *The fundamental group of  $\mathcal{D}^{2,3}$  is isomorphic to  $\mathbb{Z}$  and it is generated by  $[\alpha]$  (or by  $[\beta]$ ).*

*Proof.* This is a consequence of Proposition 3.3 and the computations in section 2:

$$\begin{array}{ccccccc} \pi_2(\mathbb{CP}^2) = \mathbb{Z}\langle[\Phi]\rangle & \xrightarrow{\delta_*^2} & \pi_1(\mathcal{D}_I^{2,2}) = \mathbb{Z}\langle[\alpha], [\beta], [\sigma]\rangle & \longrightarrow & \pi_1(\mathcal{D}^{2,2}) & \longrightarrow & 1 \\ \downarrow \cong & & \downarrow i_* & & \downarrow i_* & & \\ \pi_2(\mathbb{CP}^3) = \mathbb{Z}\langle[\Phi^3]\rangle & \xrightarrow{\delta_*^3} & \pi_1(\mathcal{D}_I^{2,3}) = \mathbb{Z}\langle[\alpha], [\beta]\rangle & \longrightarrow & \pi_1(\mathcal{D}^{2,3}) & \longrightarrow & 1 \end{array}$$

hence  $\delta_*^3([\Phi^3]) = i_*\delta_*^2([\Phi]) = i_*([\tilde{\Phi}|_{S^1}]) = i_*(2[\alpha] + 2[\beta] + [\sigma]) = [\alpha] + [\beta]$ .  $\square$

#### 4. NON PLANAR DESARGUES CONFIGURATIONS

First we analyze the fundamental group of two three-dimensional configuration spaces  $\mathcal{D}_I^3 = \mathcal{D}_I^{3,3}$  and  $\mathcal{D}^3 = \mathcal{D}^{3,3}$ .

**Lemma 4.1.** *The following projections are locally trivial fibrations:*

- a)  $\mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C}) \hookrightarrow \mathcal{D}_I^3 \rightarrow \mathcal{F}_3^{2,2}$ ,  $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (d_1, d_2, d_3)$
- b)  $\mathcal{D}_I^3 \hookrightarrow \mathcal{D}^3 \rightarrow \mathbb{CP}^3$ ,  $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto I = d_1 \cap d_2 \cap d_3$ .

*Proof.* The proofs are similar to those of Lemmas 2.1 and 2.2.  $\square$

*Proof of Theorem 1.4 (the case  $n = 3$ )*. We modify a little the previous notations: the base point in these solid Desargues configurations are related to the center  $I^0 = [0 : 0 : 1 : 0]$  and to the points:

$$\begin{aligned} A_1^0 &= [0 : 0 : 0 : 1], B_1^0 = [0 : 0 : 1 : 1], d_1^0 : X_0 = X_1 = 0, \\ A_2^0 &= [0 : 1 : 0 : 0], B_2^0 = [0 : 1 : 1 : 0], d_2^0 : X_0 = X_3 = 0, \\ A_3^0 &= [1 : 0 : 0 : 0], B_3^0 = [1 : 0 : 1 : 0], d_3^0 : X_1 = X_3 = 0. \end{aligned}$$

Using the fibrations of Lemma 4.1 we find

$$\pi_2(\mathcal{F}_3^{2,2}) \xrightarrow{\delta_*} \pi_1(\mathcal{F}_2(\mathbb{C})^3) \cong \mathbb{Z}^3 \rightarrow \pi_1(\mathcal{D}_{I^0}^3) \rightarrow 1,$$

where the first group is isomorphic with  $\pi_2(\mathcal{F}_2(\mathbb{CP}^2)) \cong \mathbb{Z}^2 = \mathbb{Z}\langle [F], [B] \rangle$  (use the fibration  $* \simeq \mathbb{CP}^2 \setminus \mathbb{CP}^1 \hookrightarrow \mathcal{F}_3^{2,2} \rightarrow \mathcal{F}_2(\mathbb{CP}^2)$ ); the homotopy classes  $[F]$  and  $[B]$  correspond to the free generators of the second homotopy groups of the fiber and of the basis respectively, in the fibration (see [3])  $\mathbb{CP}^1 \simeq (\mathbb{CP}^2 \setminus \{*\}) \hookrightarrow \mathcal{F}_2(\mathbb{CP}^2) \rightarrow \mathbb{CP}^2$ :

$$F : (D^2, S^1) \rightarrow (\mathcal{F}_3^{2,2}, d_*^0), \quad z \mapsto (d_1^0, d_2^{F(z)}, d_3^{F(z)}),$$

where  $d_2^{F(z)} : zX_0 - rX_1 = 0 = X_3$  and  $d_3^{F(z)} : rX_0 + \bar{z}X_1 = 0 = X_3$ , and also

$$B : (D^2, S^1) \rightarrow (\mathcal{F}_3^{2,2}, *), \quad z \mapsto (d_1^{B(z)}, d_2^0, d_3^{B(z)}),$$

where  $d_1^{B(z)} : zX_0 - rX_3 = 0 = X_1$ ,  $d_3^{B(z)} : rX_0 + \bar{z}X_3 = 0 = X_1$ . Choosing the lifts  $\tilde{F}, \tilde{B} : (D^2, S^1) \rightarrow (\mathcal{D}_{I^0}^3, \mathcal{F}_2(d_1^0) \times \mathcal{F}_2(d_2^0) \times \mathcal{F}_2(d_3^0))$ :

$$\tilde{F}(z) = (A_1^0, B_1^0, A_2^{\tilde{F}(z)}, B_2^{\tilde{F}(z)}, A_3^{\tilde{F}(z)}, B_3^{\tilde{F}(z)})$$

with

$$\begin{aligned} A_2^{\tilde{F}(z)} &= [r : z : 0 : 0], & B_2^{\tilde{F}(z)} &= [r : z : 1 : 0], \\ A_3^{\tilde{F}(z)} &= [\bar{z} : -r : 0 : 0], & B_3^{\tilde{F}(z)} &= [\bar{z} : -r : 1 : 0], \end{aligned}$$

respectively

$$\tilde{B}(z) = (A_1^{\tilde{B}(z)}, B_1^{\tilde{B}(z)}, A_2^0, B_2^0, A_3^{\tilde{B}(z)}, B_3^{\tilde{B}(z)})$$

with

$$\begin{aligned} A_1^{\tilde{B}(z)} &= [r : 0 : 0 : z], & B_1^{\tilde{B}(z)} &= [r : 0 : 1 : z] \\ A_3^{\tilde{B}(z)} &= [\bar{z} : 0 : 0 : -r], & B_3^{\tilde{B}(z)} &= [\bar{z} : 0 : 1 : -r], \end{aligned}$$

we obtain the equalities  $\delta_*([F]) = -[b] + [c]$ ,  $\delta_*([B]) = -[a] + [c]$ . Therefore we proved that

**Corollary 4.2.** *The fundamental group of the space  $\mathcal{D}_I^3$  is infinite cyclic generated by  $[\alpha]$ .*

Using the second fibration of Lemma 4.1, we find the exact sequence

$$\rightarrow \pi_2(\mathbb{CP}^3) \xrightarrow{\delta_*} \pi_1(\mathcal{D}_{I^0}^3) \rightarrow \pi_1(\mathcal{D}^3) \rightarrow 1$$

where the generator  $\Psi : (D^2, S^1) \rightarrow (\mathbb{CP}^3, I^0)$ ,  $z \mapsto [r : 0 : z : 0]$  has the lift

$$\tilde{\Psi} : (D^2, S^1) \rightarrow \mathcal{D}^3, \quad z \mapsto (A_1^0, B_1^{\tilde{\Psi}(z)}, A_2^0, B_2^{\tilde{\Psi}(z)}, A_3^{\tilde{\Psi}(z)}, B_3^{\tilde{\Psi}(z)}),$$

with

$$\begin{aligned} B_1^{\tilde{\Psi}(z)} &= [r : 0 : z : 1], & B_2^{\tilde{\Psi}(z)} &= [r : 1 : z : 0], \\ A_3^{\tilde{\Psi}(z)} &= [\bar{z} : 0 : -r : 0], & B_3^{\tilde{\Psi}(z)} &= [r + \bar{z} : 0 : z - r : 0]. \end{aligned}$$

Therefore  $\delta_*([\Psi]) = [\tilde{\Psi}|S^1] = [\alpha] + [\beta] + 2[\gamma] = 4[\alpha]$ , and we proved:

**Corollary 4.3.** *The fundamental group of the space  $\mathcal{D}^3$  is cyclic of order four and it is generated by  $[\alpha]$ .*

**Proposition 4.4.**

$$\begin{aligned}\pi_1(\mathcal{D}_I^{3,4}) &\cong \pi_1(\mathcal{D}_I^{3,n}) & (n \geq 4); \\ \pi_1(\mathcal{D}^{3,4}) &\cong \pi_1(\mathcal{D}^{3,n}) & (n \geq 4).\end{aligned}$$

*Proof.* This is like in 3.2. □

*Proof of Theorem 1.4 and of Theorem 1.5.* We show that  $\pi_1(\mathcal{D}_I^{3,4}) = 1$ ; this implies that  $\pi_1(\mathcal{D}^{3,4}) = 1$ . Choose as a generator for the fundamental group of the space of 3-planes in  $\mathbb{CP}^4$  containing the fixed point  $I = [0 : 0 : 1 : 0 : 0]$  the class of the map

$$\Sigma : (D^2, S^1) \rightarrow (\text{Gr}^2(\mathbb{CP}^3), X_4 = 0), \quad z \mapsto rX_1 - zX_4 = 0.$$

The lift

$$\tilde{\Sigma} : (D^2, S^1) \rightarrow (\mathcal{D}_{I_0}^{3,4}, \mathcal{D}_{I_0}^{3,3}), \quad z \mapsto (A_1^{00}, B_1^{00}, A_2^{\tilde{\Sigma}(z)}, B_2^{\tilde{\Sigma}(z)}, A_3^{00}, B_3^{00}),$$

where  $A_1^{00} = [0 : 0 : 0 : 1 : 0], \dots, B_3^{00} = [1 : 0 : 1 : 0 : 0]$  are fixed points and

$$A_2^{\tilde{\Sigma}(z)} = [0 : z : 0 : 0 : r], \quad B_2^{\tilde{\Sigma}(z)} = [0 : z : 1 : 0 : r],$$

shows that  $\delta_* : \pi_2(\text{Gr}^2(\mathbb{CP}^3)) \rightarrow \pi_1(\mathcal{D}_{I_0}^{3,3})$  is an isomorphism. □

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